Metrical Circle Map and Metrical Markov Chains

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Abstract
We propose a novel method, called Metrical Circle Map, for exploring the cyclic aspects of musical time. To this end, we give a concise formalization and introduce the notion of Metrical Markov Chains as n-th order transition probabilities of segments on the metrical circle, which leads naturally to the definition of zeroth- and first-order metrical entropy as a measure of metrical patterness and variability, and for "occupation accentuation". As a demonstration, we present an exemplary metrical analysis of five folk and pop melody collections.

1. Introduction

One important and distinctive feature of metrically bound music is the double nature of its musical time, linear on one hand, cyclic on the other. However, in most of musicological and other music-related research the focus was laid on linear aspects, while the cyclic nature of musical time was mainly investigated in the context of genuine meter and rhythm research, where rhythm and meters are occasionally illustrated using a circle representation (London, 2004; Toussaint 2004, Taslakian, 2006). But these representations were mostly limited to rhythmic patterns, e.g. clave patterns from African and Latin-American Music, or to the structural analysis of particular meters. In this paper, we like to extend and formalize this approach by introducing the Metrical Circle Map along with Metrical Markov Chains opening up several interesting possibilities of visualizing and analysing metrically bound music in a statistical way.

2. Metrical Circle Map

The events of metrically bound music are organized around underlying pulses (beats), which are grouped into higher-level units, notably the bar. The pulse is a function[1] of the sounding music, but in a certain way external to it, because it is created by listeners and performers, so it might be called "semi-autonomous". Our concept of meter will be solely based on the grouping of the pulse, in contrast to definitions incorporating accents already on a fundamental level (e.g. Temperley, 2004). Metrical accents are conceived here as a function of the grouping of the pulse (and of other musical features), playing only a secondary role. Meter can be described as a set of discrete time-points (the pulse) along with a grouping prescription. Consider a sequence of strictly monotonously increasing time-points b(i), regarded as a map from the integers into the reals. A partition of the integers is given by a mapping

$$S(k) = [i_k : i_{k+1}-1],$$

with $i_{k+1} - i_k > 0$, i.e. a collection of disjunctive intervals covering the integers.

[1] Function in the mathematical sense here.

A bar is then defined as grouping of pulses with respect to a certain partition, i.e. as the sets

$$B_k = b(S(k)) = \{ b(i_k), b(i_k +1),\ldots, b(i_{k+1}-1)\}$$

This is the basic procedure, for a full theory of meter a hierarchical procedure is needed, i.e. grouping of groupings and division of pulse, which cannot be done here due to space limitations. For the following we will only use the durations of the bars B_k given by $T_k = b(i_{k+1}) - b(i_k)$, and restrict ourselves to isochronous pulses and constant group lengths, so that the bar times T_k are all equal to a certain time T.

Consider now a rhythm conceived as an increasing sequence of time-points t_i, and an associated meter, possibly inferred from the original sequence by some beat and meter induction algorithm (e.g. Frieler, 2004), or given by manual annotation. The *Metrical Circle Map* M is then defined as a mapping from the reals into the complex unit circle S^1:

$$M_{T,\phi}(t_i) = z_i = e^{2\pi i \frac{(t_i-\phi)}{T}}$$

We have chosen the mathematical direction of counter-clockwise rotation, the complex-conjugated mapping would do the other way round.
The phase ϕ is a free parameter that can be used to gauge the map by aligning the downbeats (the beginnings of pulse groups) to a certain point on the circle. We fix for the following a gauge prescription, where the downbeats will be always aligned to the point (1,0) on the complex plane, i.e. zero phase or 3 o ‘clock.

2.1 Metrical Markov Chains

The Metrical Circle Map (MCM) lends itself quite naturally to the definition of transition probabilities between segments on the metrical circle. To this end, we define N intervals on S^1 according to

$$I_k = \{z \in S^1 | z = e^{2\pi i \frac{\phi_k}{N}}, \phi_k \in [k - \frac{1}{2}, k + \frac{1}{2}]\}$$

with 0<=k<N. The intervals cover the unit circle with the N-th roots of unity as midpoints. We can transform now a sequence of time-points into a sequence of intervals on the metrical circle, in virtue of

$$\{t_i\} \rightarrow \{I^{-1}(M_T(t_i)\}$$

where I^{-1} denotes the interval index function yielding the interval index of a point z on the circle. For these sequences of interval indices we can define Markov transition probabilities. We will restrict ourselves to zeroth- and first-order transitions

$$p(k) = p(z \in I_k), \quad p(k|j) = p(z_i \in I_k | z_{i-1} \in I_j)$$

For transition probabilities the choice of N is of course crucial. For example, for N=2 the first-order probabilities would refer to transitions between the two halves of a bar. For a more complete comparison, particularly of whole corpora with a full range of meters, a choice of N=48 seems appropriate in the most cases, which means a resolution of sextuplets in 4/4 meter or 32th notes in 3/4 meter.

Metrical Markov Chains can be visualized on a circle using the following procedure:

1. Zeroth-order probabilities are displayed by smaller circles at corresponding circle positions with radius and blackness proportional to probability
2. First-order probabilities are displayed by arrows with thickness and blackness proportional to probability.

In Figure 1 an example for a single melody metrical, the vocal line of *Mandy* by Barry Manilow. The signature is 4/4. We see a preference for events at beginnings and ends of bars with beat 2 being the most frequent position. Generally, the melody avoids the classical strongest metrical positions. The main rhythmic movement is in eighth notes with occasional 16th notes and some triplets at beat 2. There are some prominent syncopations, particularly the transitions from 4+ to 1+ and from 2+ to 3+.

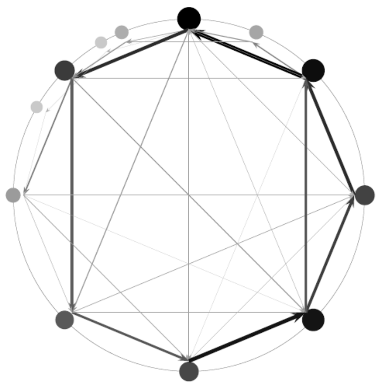

Figure 1: Vocal line of *Mandy* by Barry Manilow

Another example is shown in Figure 2. The composition "Cross-Fade" by Steve Coleman employs a considerably complex 4-bar drum groove in 9/4 time, played throughout the entire piece. Inferring the meter just by listening is even for trained musicians a highly demanding, if not impossible task. The groove is played with bass drum, snare drum, hi-hat and cowbell. The cowbell is rendering a highly asymmetrical clave pattern yielding actually a 36-beats period. The hi-hat serves as the main timekeeper playing nearly constantly on every quarter pulse. The snare drum shows a relatively simple pattern of 5 beats spread over two bars, whereby the even bars equal the odd bars plus an additional stroke on the downbeat. Finally, the bass drum repeats also only after the full 4-bar cycle, and avoids the downbeat. However, the bass drum pattern is the same in the first half of each bar with two or three varying kicks in the second half. The metrical characteristics of the groove, particularly the complexity of the cowbell pattern, are nicely reflected in the representation on the metric circle.

2.2 Metrical Entropy

Information entropy is often been used to describe stochastic distributions, which can be done for Metrical Markov Chains as well. Recall that for a discrete probability distribution of N states with probabilities p_i, Shannon's information entropy is given by the formula

$$H = -\Sigma \ p_i \log_2 p_i$$

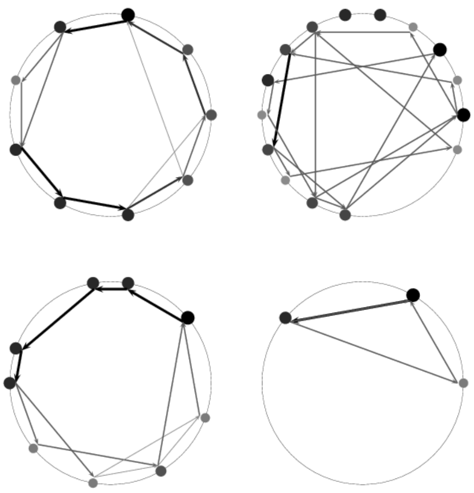

Figure 2: Single drum parts of the "Cross-Fade" groove. Top row from left to right: Hi-hat and cow bell. Bottom row from left to right: Kick and snare drum.

Information entropy measures the uncertainty of a probability distribution, which can be interpreted in terms of information (Shannon, 1948), The more confident an observer can be about the next event, the less information is provided, as nothing new (i.e. information) can be inferred from such events. The more concentrated (or “accented”) the probabilities are, the lower the entropy will be. The maximum of the entropy is reached for uniform probability distributions, where each event is equally probable.
For the matter of comparison, we normalize the entropy by dividing by the maximum entropy $H_{max}= \log_2 N$, where N is the number of possible events. Hence,

$$h = -\Sigma \; p_i \log_2 p_i / \log_2 N$$

The parameter N determines the event space and thus influences critically the entropy. Fixing an N for the Metrical Circle Map means fixing the smallest possible subdivision of a bar, but perceptually this of course depends on absolute bar length, type of the meter, and tempo. Here we encounter a general problem of statistical measures in the field of humanities, because the true event space is often not determinable, and needs a definition *a forteriori*. This means, that any interpretation of metrical entropy in terms of absolute information content is not well justified, but for the purpose of comparison it is still useful, when careful interpreted. As Feldman & Crutchfield put it: “How is the [complexity] measure to be used? What questions might it help answer?” (Feldman & Crutchfield, 1998).

For Metrical Markov Chains we define normalized zeroth- and first-order metrical entropies h_0 and h_1, where the first-order Markov Chains are regarded as probability distributions with N^2 states. What is the interpretation of these entropies then?
The zeroth-order metrical entropy h_0 becomes maximal for a uniform distribution of metrical positions. The more different metrical positions occur in a rhythm and the more similar the probabilities of these positions are, the higher the entropy will be. Conversely, the less positions appear in a rhythm and the more imbalanced the probabilities are, the lower the entropy. This could be interpreted as an indicator of “patterness” on one hand, because any constantly repeated rhythmic pattern would receive higher entropy than a rhythm with an identical set of metrical positions, but with variations in such a way, that some metrical positions are preferred over other.

Note, that form and phase of a repeated pattern does not matter. A sequence of quarter notes will always receive the same entropy, regardless how it is aligned to the beat, and any more rhythmically complex pattern, but with the same number of events, e.g. African timeline, would also gain the same zeroth-order metrical entropy. Moreover, this is also true for a rhythm jumping randomly from one metrical position to another with the number of events per bar being the same. Hence, zeroth-order metrical entropy is not fully sufficient as a measure of patterness. However, to distinguish the totally random case from the totally patterned case, the first-order entropy can to be taken into account. A totally random rhythm would have a more widespread and uniform distribution of first-order transitions, and thus higher first-order metrical entropy than a patterned rhythm, where the transitions are already completely fixed by the pattern. With the combined view of zeroth- and first order metrical entropies it is thus possible to make statements about the metrical patterness and variability of rhythms.

3. Metrical analysis of melody collections

To demonstrate the just described methods, we analysed five different song collections: 61 Irish folksongs, 586 Luxembourgian folksongs, 149 East-polish chants from Warmia, and 207 German children songs (all taken from the Essen Folksong Collection, Schaffrath, 1995), and 53 contemporary Pop songs (data kindly provided by Frank Riedemann). All songs were transformed with the MCM using an N=48 segmentation of the circle. Subsequently, zeroth- and first-order transition probabilities were calculated for every song and also accumulated over all songs.

3.1 Distribution of Signatures

The distribution of signatures in Table 1 already shows remarkably differences. The German children songs, the Warmian chants, and the Pop songs are clearly dominated by duple meters (79,2%, 72,29% and 96,2% resp.), where Children songs clearly prefer 2/4 (70,5%), and the Pop songs almost all are in 4/4 meter (only two are written in 6/4 meter). Odd meters are only found in the Warmian chants, though to a fairly small amount (2,02%). The Luxembourgian collections contains also a high share of duple meters (60,8%), with a slight preference for 2/4 over 4/4 time (33,6% vs. 27%). The Irish songs are insofar distinguished from each other collection, as the share of triple and compound duple meters is higher (56%) than the share of duple meters (44%).

Signature	Children	Warmia	Luxembourg	Irish	Pop
2/4	70,5%	4,05%	33,6%	11%	
4/8			0,2%		
4/4	8,7%	72,29%	27%	33%	96,2%
8/4		2,02%			
3/8	3,9%	1,35%	2,0%		
3/4	12,1%	7,43%	21,5%	33%	
6/4		1,35%		3%	3,8%
6/8	4,8%	1,35%	15,7%	16%	
9/8				3%	
9/4		8,1%			

5/4		1,35%			
7/4		0,67%			
Total duple	79,2%	78,36%	60,8%	44%	96,2%
Total triple	20,8%	19,58%	39,2%	56%	3,8%
Total odd		2,02%			

Table 1: Distribution of signatures in the melody collections

3.2 Analysis of Metrical Markov Chains

The joint visualisations for zeroth- and first-order Metrical Markov Chains of the five melody collections can be found in Figures 3-7. As described above, the size and blackness of the small circles are proportional to the occupation probabilities, and the thickness and blackness of the arrows connecting two metric positions is proportional to the first-order transition probability. The downbeat lies at 3 o'clock, and time is running counter-clockwise.

By looking on the graphs, the squares and octagons from duple, and the triangles and hexagons from compound duple and triple meters appear as the most prominent shapes. The distributions of signatures in the collection is clearly reflected too, particularly the mixture of duple and triple meters in the Irish collection, and the smaller share of triple meters in the Children, Warmian and Pop songs. In all collections, the most frequent transitions are constituted by quarter- and eighth notes movement, which is of course not surprising. For the Pop songs a certain lack of quarter note movement along the main beats can be stated, however, we can find a rotated square here, stemming from syncopations. A further comparison with the other diagrams reveals that syncopations are almost exclusively present in the Pop song collection.

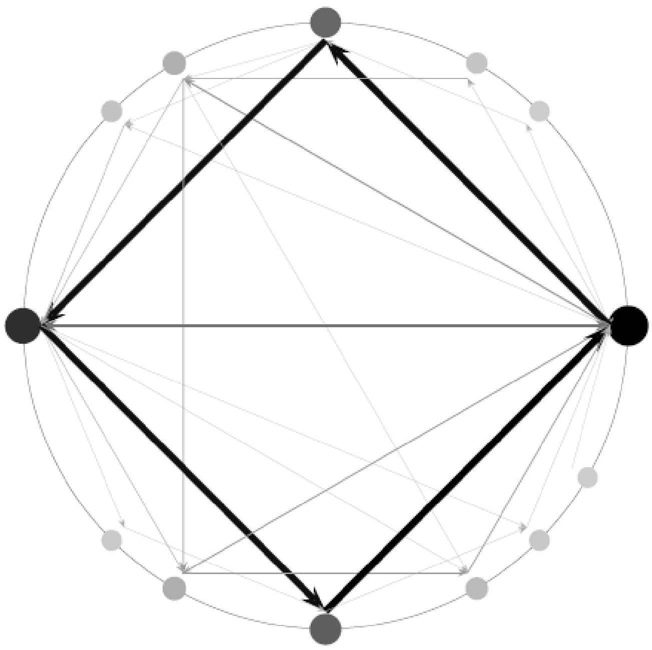

Fig. 3: Metrical Markov probabilities of German Children songs (N=207)

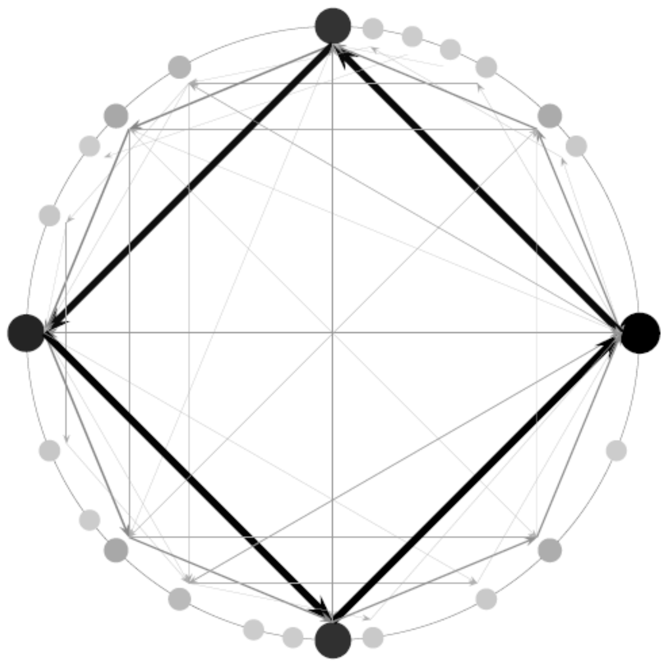

Fig. 4: Metrical Markov probabilities of Warmian chants (N=190)

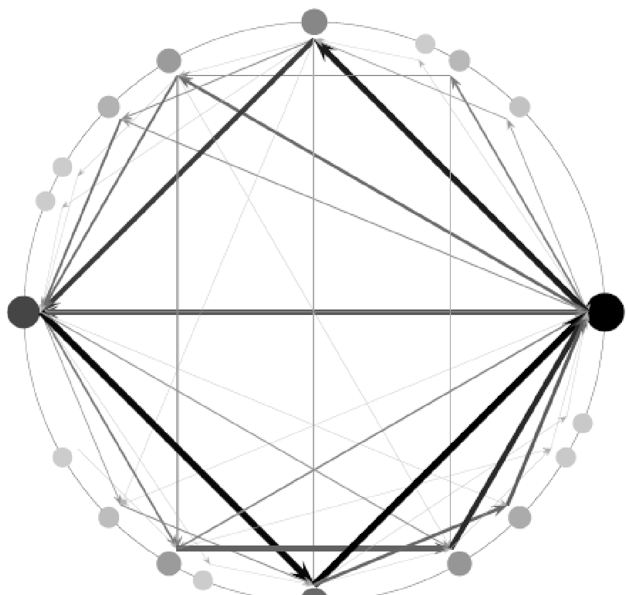

Fig. 5: Metrical Markov probabilities of Luxembourgian folk songs (N=586)

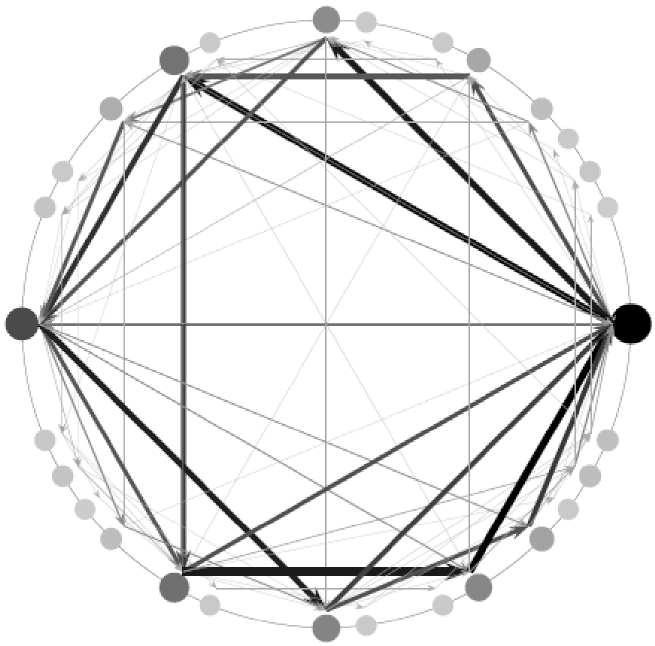

Fig. 6: Metrical Markov probabilities of Irish folk tunes (N=61)

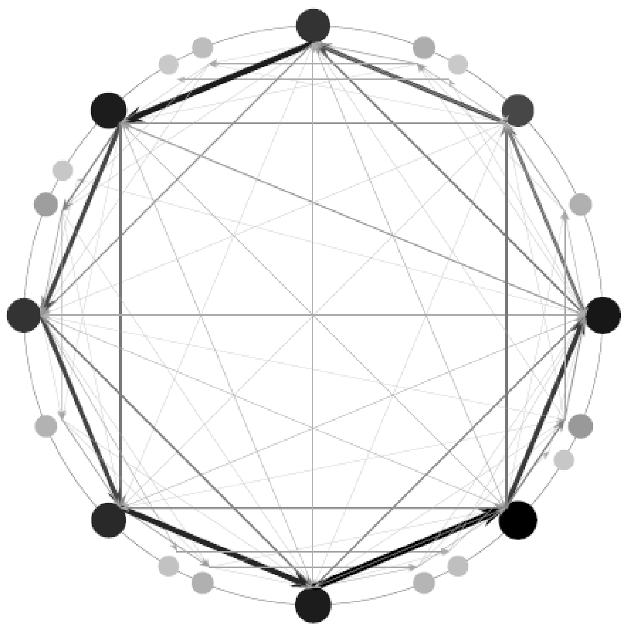

Fig. 7. Metrical Markov probabilities of the Pop songs (N=53)

An alternative display of the distribution of metrical positions is shown in Figure 9. Here the relative frequencies of metrical positions of all collections are plotted against the indices on the Metrical Circle. The downbeat is the most frequent position in every collection except one: the Pop songs. Here the anticipated one (4+) appears slightly more often than the downbeat. Similarly, the anticipated 3 (2+) occurs more frequent than the third beat itself. Generally, in the Pop songs every eighth-note position is quite equally probable indicating a high rhythmic and metric variability of the vocal lines, or, connected to this, a high amount of syncopation. The Children songs and the Warmian chants show distributions, which could be expected in the light of western metrical theory, and what might be termed "occupation accentuation". The classical hierarchy of metrical positions is nicely reflected in the occupation probabilities. The rank order of beats in 4/4 meter is then 1, 3, 4, 2, (or 1+, 2, 2+, 1+ in 2/4 time).[2] Similar statements for the Luxembourg and Irish collections hold as well as will be corroborated by the analysis of metrical entropies below.

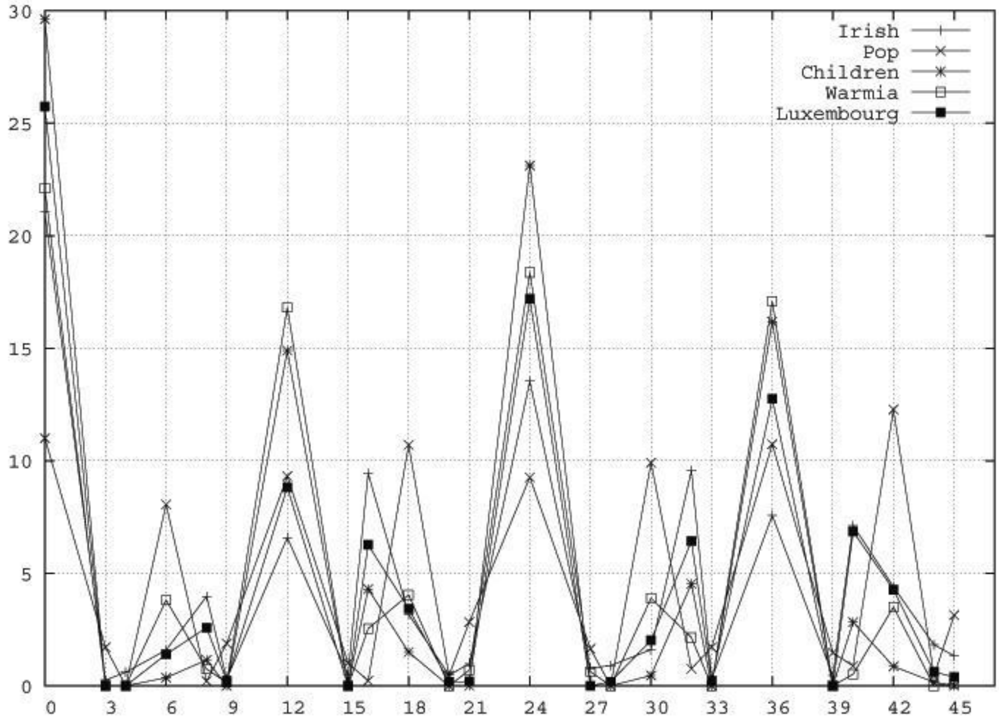

Fig. 9: Frequencies of metric positions with N=48 for the five melody collections.

An alternative display of first-order Metrical Markov Chains can be found in Fig. 10. The transition probabilities form an asymmetric 48x48 Matrix. The size of the bubbles at each grid point in the graph is proportional to the value of the corresponding matrix entry. The quarter- and eighth-note movement is reflected in the concentration of bubbles on the upper secondary diagonals. The lower right triangle contains much less bubbles, which means a lack of metrical transitions across bar borders. An exception is the concentration of bubbles on the x-axis, where the transitions to the downbeat can be found, which generally belong to the class of most frequent transitions. Another exception is the column of bubbles above index 42 (4+ in 4/4 time) on the outer right, belonging to the Pop songs. A similar column can be found above metrical position 6 (1+) in the upper left corner, also from the Pop songs.

[2] Looking at the German children songs this would suggest, that the its most common 2/4 signature should be in fact written as 4/8.

These transitions are due to phrase endings at the anticipated 1 with subsequent phrases beginning somewhere in the next bar, or due to transitions across the bar, touching not the downbeat but the delayed downbeat on 1+, as for example can be seen in *Mandy.* This is another indication for the high amount of syncopation in Pop songs.

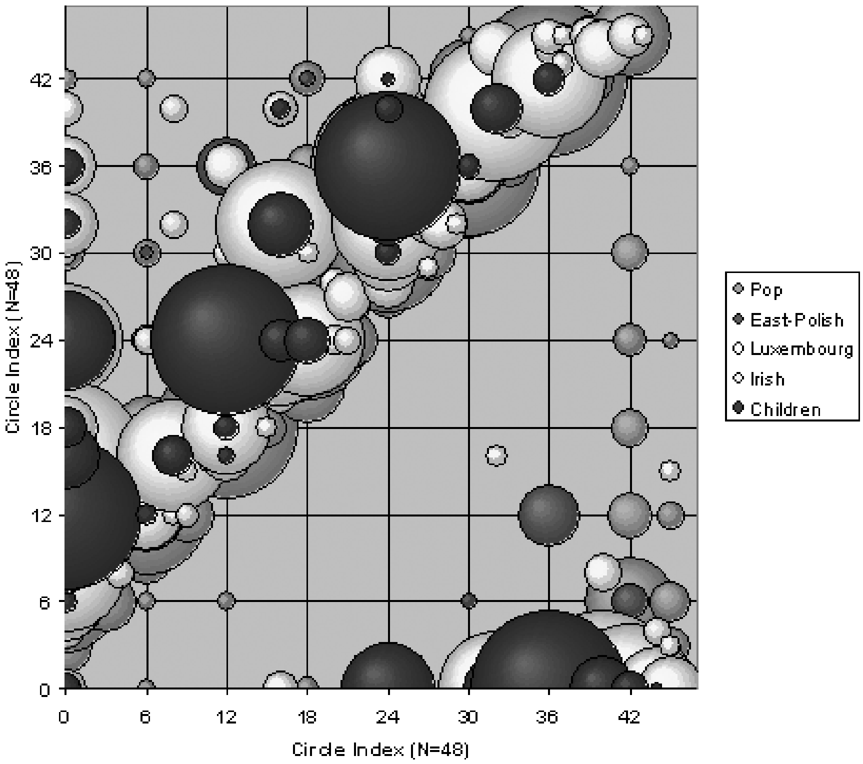

Fig. 10: Metrical Markov first-order transition probabilities.

	Children	Warmia	Luxembourg	Irish	Pop
h_0	0.38	0.44	0.43	0.48	0.59
h_1	0.24	0.28	0.27	0.31	0.40
σ_{avg}	0.031	0.059	0.047	0.065	0.05

Table 2: Means and averaged standard deviation of metrical entropies of the five song collections

As explained above, metrical variability can be measured using the zeroth- and first-order metrical entropies. Mean values and mean standard deviation are shown in Table 2. Boxplots of the entropy distributions are depicted in Figure 11. The metrical most homogenous collection of the most “accented” songs are the German Children songs. The Warmian and Luxembourgian tunes show quite similar distributions to each other with quite low average values indicating that these songs are also quite “occupation accented”. The Pop songs have the highest metrical entropies, which confirms the observations from above, that the Pop songs have a close to uniform distribution of eighth-notes positions and the highest spread of metrical transitions. The Irish songs are somewhere in between, but more close to the folk songs with regard to average values, thus also showing metrical “occupation accentuation”. However, they possess the highest variance of all distributions, stemming partly from the mixture of signatures here. On the opposite side, the most homogenous collections are the Children and the Luxembourgian songs, closely followed by the Pop songs and the Warmian songs.

To check, whether the distributions of metrical entropies are statistically different, we employed Welch’s t-test and found all differences between all distributions of metrical entropies to be highly significant (p<0.00), except for the entropies of the Warmian and the Luxembourgian songs. A conclusion that can be drawn is, that metrical entropies can serve as distinction criterion between different genres and folksong styles, at least on the level of collections. And we believe, that in conjunction with other melodic and rhythmic features, classification of single songs should be possible as well.

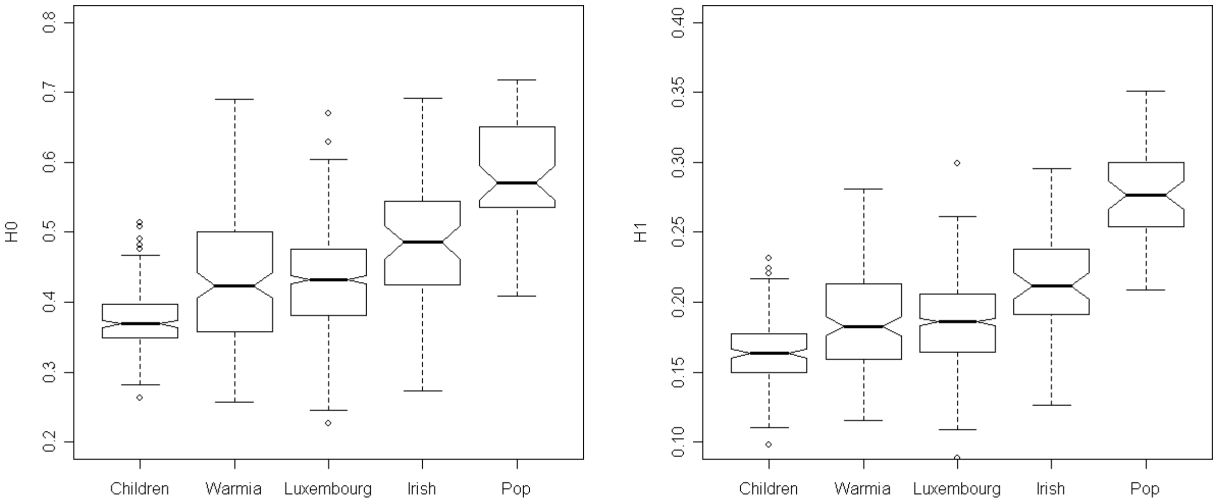

Fig. 11: Boxplots of metrical entropies. Left: Zeroth-order. Right: First-order entropies.

4. Conclusion & Outlook

We presented the methods of Metrical Circle Map and Metrical Markov Chains. Using metrical occupation and transition probabilities, melodies and entire corpora, as well as polyphonic music, can be analyzed and instructively visualized, thus giving deeper insights into the cyclic organization of music and in the metrical peculiarities of different music and styles.
Several extensions to the Metrical Circle Map, incorporating other dimensions of music, can be imagined. The pitch dimension of melodies could be reflected, for instance, in the radial dimension of the circle. By using a circle of fifth representation of pitch, we would arrive at toroidal representations of melodies. Linear time could be used as an additional coordinate as well, resulting in spirals on metrical cylinders. All these geometrical representations might possibly be used for similarity algorithms as well. Furthermore, drawing on distance measures for probability distribution, e.g. the Jensen-Shannon divergence, similarity measures for Metrical Markov Chains could be constructed as well.
We hope to explore some of these possibly fruitful extensions in the future.

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